

Dynamics of random chains of finite size with an infinite number of elements in \mathbb{R}^{2*}

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Abstract

This article studies the dynamics of a finite chain with infinite components. The equation which permits us to find the probability distribution of the chain length is constructed and analysed. This research is a continuation of paper [1].

1 Formulation of the problem

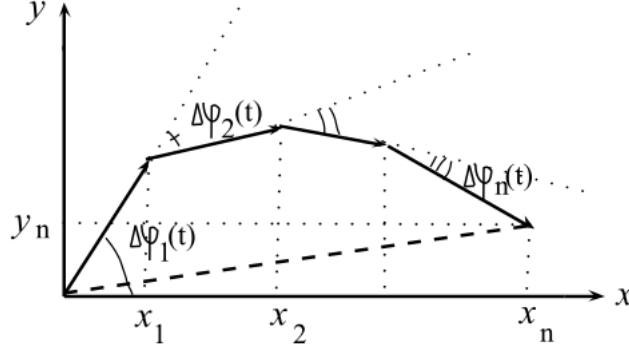
In Feller's book [2] the problem of the length of a random chain is considered, this chain is described in the following way: the number of the elements is equal to n , the length of all its elements is equal to one, the angle of one component with respect to the previous is always the same up to a sign (the probability of each angle is equal to $1/2$), the distance between the end points of the chain (length of the chain) is defined by means of the average square length

$$\mathbf{M}[L_n^2] = n \frac{1 + \cos \alpha}{1 - \cos \alpha} - 2 \cos \alpha \frac{1 - \cos^n \alpha}{(1 - \cos \alpha)^2}.$$

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We will consider the following chain: the length of the chain is finite, the number of the components is infinite, the length of each component is a random variable, the angle of each component with respect to the previous one is also random.



The physical model can be a rope in a medium of Brownian particles, as length of the chain we can understand the modulus of the vector joining the starting point and the end point of the chain.

Let $l \in [0, L]$ a parameter, L a constant, l_1, l_2, \dots the values of the parameter, $l_1 < l_2 < \dots \leq L$, $\Delta = L/N$, $l_j = j \cdot \Delta$. We will consider the model of the chain described by the following system of equations:

$$x_N(t) = \sum_{s=1}^N a(l_s) \Delta \cdot \cos \varphi_s(t), \quad y_N(t) = \sum_{s=1}^N a(l_s) \Delta \cdot \sin \varphi_s(t), \quad (1)$$

where $a(l_s), \varphi_s(t)$ in general are random processes, $a(l) > 0$, $\varphi_s(t)$ is the angle between the s -th component with respect to the previous one, $\varphi_1(t)$ is the angle of the first component of the chain with respect to the positive direction of the x -axis.

If we denote by $|x_n(t)|^2 + |y_n(t)|^2$ the length of the chain consisting of the n elements, then the length $\Delta(l)$ of the real component of the chain is expressed by the variable

$$\Delta(l) = a(l) \Delta, \quad a(l) > 0, \quad \int_0^L a(l) dl = \mathcal{L}.$$

Models of type (1) describe the distribution of the length $\mathcal{L}(t)$ of the chain for the case where the following inequality is satisfied:

$$\mathcal{L}^2(t) = |x_N(t)|^2 + |y_N(t)|^2 \leq \text{const.}$$

From the point of view of the representation of the phenomenon of the turbulent diffusion, the model (1) can be useful for some generalizations of the passive displacement under the action of vortices of different size [3].

Let be $n < N$ (that is we will consider not the whole chain but a part of it), $N \rightarrow \infty$. Since the coordinates of the initial point and the end point of each component depend on time t and from the parameter l , then in the model (1) we introduce some changes.

$$x_n(l; t) = \sum_{s=1}^n a(l_s) \Delta \cdot \cos \varphi_s(t), \quad y_n(l; t) = \sum_{s=1}^n a(l_s) \Delta \cdot \sin \varphi_s(t). \quad (2)$$

In this way, the random field $\{x_n(l; t); y_n(l; t)\}$ is a dynamical stochastic process. We will study its limit behavior for $n \rightarrow \infty$.

2 Assumptions on the model

In order to obtain the coefficient of the limit equation in analytical form we shall restrict ourself to the model satisfying the following assumptions:

$$\begin{aligned} a(l) &> 0, \quad l \in [0, L], \\ \varphi_s(t) &= \sum_{k=1}^s \eta(l_k; t) \Delta(w(l_k)), \quad t \in [0, T], \\ \eta(l_k; t) &= \int_0^t \sigma(l_k; \tau) dw_k(\tau), \end{aligned} \quad (3)$$

where $\Delta(w(l_k))$, $\Delta(w_k(\tau))$ are independent among themselves and for different s and τ are anticipating increments of the corresponding Wiener processes defined on the product of independent probability spaces

$$\{\Omega_1, \mathfrak{F}_l, P_1\} \times \{\Omega_2, \mathfrak{F}_t(n), P_2\},$$

where \mathfrak{F}_l , and $\mathfrak{F}_t(n)$ are the corresponding flows of sigma algebras generated by the processes $w(l)$ and $w(t) \in \mathbb{R}^n$; the functions $a(l) \in \mathbb{C}_{[0, L]}^1$ and $\sigma(l; t) \in \mathbb{C}_{[0, L] \times [0, T]}^2$ are deterministic functions depending on l and t , $\eta(l_s; t)$ is the intensity of the angle.

We have therefore,

$$\begin{aligned} x_n(l; t) &= \sum_{s=1}^n a(l_s) \Delta \cdot \cos \left[\sum_{k=1}^s \left(\int_0^t \sigma(l_k; \tau) dw_k(\tau) \right) \Delta(w(l_k)) \right] \\ y_n(l; t) &= \sum_{s=1}^n a(l_s) \Delta \cdot \sin \left[\sum_{k=1}^s \left(\int_0^t \sigma(l_k; \tau) dw_k(\tau) \right) \Delta(w(l_k)) \right] \end{aligned} \quad (4)$$

Under the condition of bounded length of the chain for the random function $\varphi_s(t)$ can be defined the limit for $n \rightarrow \infty$. In this context the variable l appears as a parameter.

3 Transition to auxiliary processes

Let us transform (2) by means of the Euler representation:

$$\begin{aligned} x_n(l; t) &= \sum_{s=1}^n a(l_s) \Delta \cdot \cos \varphi_s(t) = \sum_{s=1}^n a(l_s) \Delta \cdot \frac{\exp\{i\varphi_s(t)\} + \exp\{-i\varphi_s(t)\}}{2} = \\ &= \frac{1}{2} \sum_{s=1}^n a(l_s) \Delta \cdot \exp\{i\varphi_s(t)\} + \frac{1}{2} \sum_{s=1}^n a(l_s) \Delta \cdot \exp\{-i\varphi_s(t)\}, \\ y_n(l; t) &= \sum_{s=1}^n a(l_s) \Delta \cdot \sin \varphi_s(t) = \sum_{s=1}^n a(l_s) \Delta \cdot \frac{\exp\{i\varphi_s(t)\} - \exp\{-i\varphi_s(t)\}}{2i} = \\ &= \frac{1}{2i} \sum_{s=1}^n a(l_s) \Delta \cdot \exp\{i\varphi_s(t)\} - \frac{1}{2i} \sum_{s=1}^n a(l_s) \Delta \cdot \exp\{-i\varphi_s(t)\}, \end{aligned}$$

we now introduce the auxiliary process

$$\begin{aligned} z_1(s; t) &= \exp\left\{-i \sum_{j=1}^s \Delta w(l_j) \int_0^t \sigma(l_j; \tau) dw_j(\tau)\right\}, \\ z_{n,1}(l, t) &= \sum_{s=1}^n a(l_s) \Delta \cdot \exp\{-i\varphi_s(t)\} = \sum_{s=1}^n a(l_s) \Delta \cdot z_1(s; t), \quad \Delta = O(n^{-1}). \end{aligned}$$

By using the Euler representation we rewrite the process $\{x_n(l; t); y_n(l; t)\}$ in the following form

$$x_n(l; t) = \frac{1}{2}(z_{n,1}(l, t) + z_{n,1}^*(l, t)), \quad y_n(l; t) = \frac{i}{2}(z_{n,1}(l, t) - z_{n,1}^*(l, t)).$$

For the construction of the characteristic function of the random field $\{x_n(l; t); y_n(l; t)\}$ we define the form of the function $\exp\{i(\alpha x_n(l; t) + \beta y_n(l; t))\}$:

$$\begin{aligned} \exp\{i(\alpha x_n(l; t) + \beta y_n(l; t))\} &= \exp\left\{i\alpha \frac{z_{n,1}(l; t) + z_{n,1}^*(l; t)}{2} - \beta \frac{z_{n,1}(l; t) - z_{n,1}^*(l; t)}{2}\right\} = \\ &= \sum_{m,r=1}^{\infty} \frac{(i\alpha - \beta)^m (i\alpha + \beta)^r}{2^{m+r} m! r!} z_{n,1}^m(l; t) z_{n,1}^{*r}(l; t). \end{aligned}$$

In consequence the analysis of the process $\{x_n(l; t); y_n(l; t)\}$ leads to the study of the process $z_{n,1}^m(l; t) z_{n,1}^{*r}(l; t)$.

Since the summation and the integration operations have the same properties we replaced (in symbolic form, when $\Delta \rightarrow 0$ and this corresponds to $n \rightarrow \infty$) the process

$$z_{n,1}(l, t) = \sum_{s=1}^n a(l_s) \Delta \cdot \exp\left\{-i \sum_{j=1}^s \left(\int_0^t \sigma(l_j; \tau) dw_j(\tau)\right) \Delta w(l_j)\right\}$$

by the process

$$z_{,1} = \sum_{s=1}^n a(l_s) \Delta \cdot \exp\left\{-\sum_{j=1}^s \eta(l_s, t) \Delta w(l_s)\right\} = \int_0^l a(u) \exp\left\{-i \int_0^u \eta(\theta, t) dw(\theta)\right\} du,$$

where $\eta(u, t) = \int_0^t \sigma(u, \tau) d\tau$. We do not loose any generality in the analysis

with this assumption and in the sequel we shall use the symbol \int instead of \sum .¹

4 Degree transformation

By considering the continuity of the process $z_{n,1}(l; t)$ and, consequently, of the process $z_{,1}(l; t)$ with respect to both variables l and t , we produce the

¹This symbol does not concern known designations. It is a label only.

degree transformation:

$$\begin{aligned}
z_{,1}^m(l; t) &= \left[\int_0^l a(u) \exp \left\{ -i \int_0^u \eta(\theta; t) dw(\theta) \right\} du \right]^m = \\
&= m! \int_0^l a(u_1) du_1 \exp \left\{ -i m \int_0^{u_1} \eta(\theta_1; t) dw(\theta_1) \right\} \times \\
&\times \int_{u_1}^l a(u_2) du_2 \exp \left\{ -i(m-1) \int_{u_1}^{u_2} \eta(\theta_2; t) dw(\theta_2) \right\} \times \dots \times \\
&\times \int_{u_{m-1}}^l a(u_m) du_m \exp \left\{ -i \int_{u_{m-1}}^{u_m} \eta(\theta_m; t) dw(\theta_m) \right\}
\end{aligned} \tag{5}$$

where $0 < u_1 < \dots < u_m < l$. In different intervals we have different $dw(\theta)$ for each time instant t .

5 Determination of moments

Since the process $z_{,1}^m(l; t) z_{,1}^{*r}(l; t)$ depends from two variables, for the calculation of the mean $\mathbf{M} [z_{,1}^m(l; t) z_{,1}^{*r}(l; t)]$ it is necessary to carry out the averaging for t constant (on the space Ω_1), and then Ω_l .

5.1 Averaging with respect to l

On the disjoint intervals $[u_i, u_{i+1})$ for all i , for all i the processes

$$f(u_i, u_{i+1}) = \int_{u_i}^{u_{i+1}} \eta(\theta_{i+1}; t) dw(\theta_{i+1})$$

are independent by construction (since $dw(\theta)$ are Wiener processes, $\eta(\theta; t)$ for fixed t is a non-random function depending on θ). Because of this, since $a(l)$ is also a non-random function, then the mathematical mean of each factor is defined in the following way:

$$\begin{aligned}
\mathbf{M}_t [z_{,1}^m(l; t)] &= m! \int_0^l a(u_1) du_1 \mathbf{M}_t \left[\exp \left\{ -i m \int_0^{u_1} \eta(\theta_1; t) dw(\theta_1) \right\} \right] \times \\
&\times \int_{u_1}^l a(u_2) du_2 \mathbf{M}_t \left[\exp \left\{ -i(m-1) \int_{u_1}^{u_2} \eta(\theta_2; t) dw(\theta_2) \right\} \right] \times \dots \times \\
&\times \int_{u_{m-1}}^l a(u_m) du_m \mathbf{M}_t \left[\exp \left\{ -i \int_{u_{m-1}}^{u_m} \eta(\theta_m; t) dw(\theta_m) \right\} \right].
\end{aligned} \tag{6}$$

In this way it is necessary to find the mean of the expression of the following type:

$$\exp \left\{ -i(m-j) \int_{u_j}^{u_{j+1}} \eta(\theta; t) dw(\theta) \right\}.$$

Lemma 5.1 *The following equality holds*

$$\mathbf{M}_t \left[\exp \left\{ \alpha \int_a^b \eta(u; t) dw(u) \right\} \right] = \exp \left\{ \frac{1}{2} \alpha^2 \int_a^b \eta^2(u; t) du \right\}. \quad (7)$$

Proof. Let us denote:

$$q(a, b; t) = \int_a^b \eta(u; t) dw(u) \quad (8)$$

and differentiate $q(a, b; t)$ with respect to the upper limit b . As a result we obtain:

$$d_b q(a, b; t) = \eta(b; t) dw(b).$$

Therefore, by Ito formula, the stochastic differential with respect to b of the expression

$$\exp \left\{ \alpha \int_a^b \eta(u; t) dw(u) \right\} = \exp \{ \alpha q(a, b; t) \}$$

is equal to

$$d_b \exp \{ \alpha q(a, b; t) \} = \exp \{ \alpha q(b; t) \} \alpha \eta(b; t) dw(b) + \frac{1}{2} \alpha^2 \eta^2(b; t) \exp \{ \alpha q(a, b; t) \} db.$$

We compute the average with respect to l of the obtained expression:

$$d_b \mathbf{M}_t [\exp \{ \alpha q(a, b; t) \}] = \frac{1}{2} \alpha^2 \eta^2(b; t) \mathbf{M}_t [\exp \{ \alpha q(a, b; t) \}] db. \quad (9)$$

Let us denote:

$$I_1(a, b; t) = \mathbf{M}_t [\exp \{ \alpha q(a, b; t) \}]. \quad (10)$$

Let $\eta(b; t)$ be independent from the stochastic process $w(u)$. In view of (9) we obtain the differential equation

$$\frac{dI_1(a, b; t)}{db} = \frac{1}{2} \alpha^2 \eta^2(b; t) I_1(a, b; t)$$

and its solution

$$I_1(a, b; t) = \exp \left\{ \frac{1}{2} \alpha^2 \int_a^b \eta^2(u; t) du \right\},$$

satisfies the initial condition $I_1(a, a; t) = 1$. In view of (8), (10) the statement of the lemma is proved. \square

As a consequence of Lemma 5.1, the mathematical mean (6) takes the form (for t constant):

$$\begin{aligned} \mathbf{M}_t [z_{,1}^m(l; t)] &= m! \int_0^l a(u_1) du_1 \exp \left\{ -\frac{m^2}{2} \int_0^{u_1} \eta^2(\theta; t) d\theta \right\} \times \\ &\times \int_{u_1}^l a(u_2) du_2 \exp \left\{ -\frac{(m-1)^2}{2} \int_{u_1}^{u_2} \eta^2(\theta; t) d\theta \right\} \times \dots \times \\ &\times \int_{u_{m-1}}^l a(u_m) du_m \exp \left\{ -\frac{1}{2} \int_{u_{m-1}}^{u_m} \eta^2(\theta; t) d\theta \right\}. \end{aligned}$$

5.2 Averaging with respect to t

Now we make the averaging of the process $z_{,1}^m(l; t)$ on the space Ω_2 . Since $w_s(t)$ for all s are independent Wiener processes, then $\eta^2(l_s; t)$ are independent, the average of the product is therefore equal to the product of the means. As a result we obtain:

$$\begin{aligned} \mathbf{M} [\mathbf{M}_t [z_{,1}^m(l; t)]] &= m! \int_0^l a(u_1) du_1 \mathbf{M} \left[\exp \left\{ -\frac{m^2}{2} \int_0^{u_1} \eta^2(\theta; t) d\theta \right\} \right] \times \\ &\times \dots \int_{u_{m-1}}^l a(u_m) du_m \mathbf{M} \left[\exp \left\{ -\frac{1}{2} \int_{u_{m-1}}^{u_m} \eta^2(\theta; t) d\theta \right\} \right]. \end{aligned}$$

Lemma 5.2 *The following relationship holds*

$$\mathbf{M} \left[\exp \left\{ -\frac{\alpha^2}{2} \int_a^b \eta^2(u; t) du \right\} \right] = \exp \left\{ -\frac{\alpha^2}{4} \int_a^b \left(\int_0^t \sigma^2(u; \tau) d\tau \right) du \right\},$$

where

$$\eta(u; t) = \int_0^t \sigma(u; \tau) dw(\tau),$$

$\sigma(u; t)$ is a non-random function.

Proof. We shall use the following representation:

$$\int_a^b \eta^2(u; t) du = \frac{b-a}{N} \sum_{k=1}^N \eta^2(u_k; t),$$

this is possible in force of the model assumptions. The processes $\eta^2(u_k; t)$ ($k = \overline{1, N}$) by definition are independent for different values of k . We introduce the notation:

$$P_k(t) = \eta(u_k; t) = \int_0^t \sigma(u_k; \tau) dw_k(\tau),$$

where $\sigma(u_k; t)$ is a non-random function depending on u_k, t . We consider now two cases.

A. Let $\sigma(u_k; t)$ be constant. For the seek of simplicity in the sequel we assume that $\sigma(u_k; t) = 1$ and study the problem for the processes

$$\eta(u_k; t) = \int_0^t dw_k(\tau) = \tilde{P}_k(t). \quad (11)$$

By considering the representation of the integral in form of sums, we carry out the transformation:

$$\exp \left\{ -\frac{\alpha^2(b-a)}{2N} \eta^2(u_k, t) \right\} = \exp \left\{ -\frac{\alpha^2(b-a)}{2N} \left[\int_0^t dw_k(\tau) \right]^2 \right\} = \exp \left\{ -\frac{\alpha^2(b-a)}{2N} \tilde{P}_k^2(t) \right\}.$$

Therefore

$$\exp \left\{ -\frac{\alpha^2(b-a)}{2N} \sum_{k=1}^N \tilde{P}_k^2(t) \right\} = \prod_{k=1}^N \exp \left\{ -\frac{\alpha^2(b-a)}{2N} \tilde{P}_k^2(t) \right\}$$

We denote by:

$$\exp \left\{ -\frac{\alpha^2(b-a)}{2N} \sum_{k=1}^N \tilde{P}_k^2(t) \right\} = I_N(k, \alpha^2).$$

Since

$$\mathbf{M} [\exp \{\alpha q(a, b; t)\}] = \mathbf{M} [\exp \{-\alpha q(a, b; t)\}],$$

and the following relationships hold:

$$\begin{aligned} & \mathbf{M} \left[\prod_{k=1}^N \exp \left\{ -\frac{\alpha^2(b-a)}{2N} \tilde{P}_k^2(t) \right\} \right] = \\ & = \prod_{k=1}^N \mathbf{M} \left[\exp \left\{ -\frac{\alpha^2(b-a)}{2N} \tilde{P}_k^2(t) \right\} \right] \xrightarrow{qm} \mathbf{M} [\exp \{ \alpha q(a, b; t) \}]. \end{aligned}$$

Then we carry out the Ito differentiation:

$$d\tilde{P}_k(t) = dw_k(t),$$

$$d\tilde{P}_k^2(t) = dt + 2w_k(t) dw_k(t) \quad (12)$$

and in view of (11) we have that:

$$\begin{aligned} d_t \exp \left\{ -\frac{\alpha^2(b-a)}{2N} \tilde{P}_k^2(t) \right\} &= -\exp \left\{ -\frac{\alpha^2(b-a)}{2N} \tilde{P}_k^2(t) \right\} \frac{\alpha^2(b-a)}{2N} d\tilde{P}_k^2(t) + \\ &+ \frac{\alpha^4(b-a)^2}{2N^2} \tilde{P}_k^2(t) \exp \left\{ -\frac{\alpha^2(b-a)}{2N} \tilde{P}_k^2(t) \right\} dt. \end{aligned}$$

We introduce in the last differential the expression (12):

$$\begin{aligned} d_t \exp \left\{ -\frac{\alpha^2(b-a)}{2N} \tilde{P}_k^2(t) \right\} &= -\exp \left\{ -\frac{\alpha^2(b-a)}{2N} \tilde{P}_k^2(t) \right\} \times \\ &\times \left[\frac{\alpha^2(b-a)}{2N} 2\tilde{P}_k(t) dw_k(t) - \frac{\alpha(b-a)}{2N} \left(-1 + \frac{\alpha^2(b-a)}{N} \tilde{P}_k^2(t) \right) dt \right]. \end{aligned}$$

We calculate the mean for the last expression by denoting

$$I_2(t; \alpha^2) = \mathbf{M} \left[\exp \left\{ -\frac{\alpha^2(b-a)}{2N} \tilde{P}_k^2(t) \right\} \right].$$

We obtain the equation:

$$dI_2(t; \alpha^2) = I_2(t; \alpha^2) \frac{\alpha^2(b-a)}{2N} dt + \frac{\alpha^4(b-a)^2}{2N^2} \mathbf{M} \left[\tilde{P}_k^2(t) \exp \left\{ -\frac{\alpha^2(b-a)}{2N} \tilde{P}_k^2(t) \right\} \right] dt.$$

By considering the differentiation with respect to α^2 of the expression

$$\exp \left\{ -\frac{\alpha^2(b-a)}{2N} \tilde{P}_k^2(t) \right\},$$

the last equation can be represented as a partial differential equation with constant coefficients:

$$\frac{dI_2(t; \alpha^2)}{dt} = -\frac{\alpha^2(b-a)}{2N} I_2(t; \alpha^2) - \frac{\alpha^4(b-a)}{N} \frac{\partial}{\partial \alpha^2} I_2(t; \alpha^2). \quad (13)$$

The solution of this equation will be obtained by exploiting the properties of the stochastic processes. With this purpose we evaluate the mean of the function

$\exp \left\{ -\frac{\alpha^2(b-a)}{2N} \tilde{P}_k^2(t) \right\}$. By considering that the process is a Wiener process (that is a Gaussian process) we have that:

$$\begin{aligned} \mathbf{M} \left[\exp \left\{ -\frac{\alpha^2(b-a)}{2N} \tilde{P}_k^2(t) \right\} \right] &= \int_{-\infty}^{\infty} \exp \left\{ -\frac{\alpha^2(b-a)}{2N} x^2 \right\} \times \frac{1}{\sqrt{2\pi t}} \exp \left\{ -\frac{x^2}{2t} \right\} dx = \\ &= \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{\infty} \exp \left\{ -\left(\frac{\alpha^2(b-a)}{2N} x^2 + \frac{x^2}{2t} \right) \right\} dx. \end{aligned}$$

Furthermore

$$\mathbf{M} \left[\exp \left\{ -\frac{\alpha^2(b-a)}{2N} \tilde{P}_k^2(t) \right\} \right] = \frac{1}{\sqrt{2\pi t}} \cdot \sqrt{2\pi} \cdot \sqrt{t / \left(\frac{\alpha^2(b-a)}{2N} t + 1 \right)} = \left(\frac{\alpha^2(b-a)}{2N} t + 1 \right)^{-1/2}.$$

In this way, the obtained expression $I_2(t; \alpha^2) = \left(\frac{\alpha^2(b-a)}{2N} t + 1 \right)^{-1/2}$ is the solution of the differential equation (13). Besides this, in view of Lemma 5.1, we have that:

$$\begin{aligned} \mathbf{M}_t [\exp \{ -\alpha q(a, b; t) \}] &= \exp \left\{ -\frac{\alpha^2}{2} \int_a^b \eta^2(u, t) \right\} = \exp \left\{ -\frac{\alpha^2(b-a)}{2N} \sum_{k=1}^N \eta^2(u_k, t) \right\} = \\ &= \prod_{k=1}^N \exp \left\{ -\frac{\alpha^2(b-a)}{2N} \eta^2(u_k, t) \right\}. \end{aligned}$$

Therefore

$$\begin{aligned} \mathbf{M} [\mathbf{M}_t [\exp \{ -\alpha q(a, b; t) \}]] &\stackrel{qm}{\rightarrow} \prod_{k=1}^N \mathbf{M} \left[\exp \left\{ -\frac{\alpha^2(b-a)}{2N} \tilde{P}_k^2(t) \right\} \right] = \\ &= \prod_{k=1}^N \left(\frac{\alpha^2(b-a)}{2N} t + 1 \right)^{-1/2} = \left(\frac{\alpha^2(b-a)}{2N} t + 1 \right)^{-N/2} \end{aligned}$$

Passing to the limit we obtain the complete averaging with respect to both components:

$$\mathbf{M} [I_1(a, b; t)] = \lim_{N \rightarrow \infty} \left(\frac{\alpha^2(b-a)}{2N} t + 1 \right)^{-N/2} = \exp \left\{ -\frac{\alpha^2(b-a)t}{4} \right\}.$$

B. Now we consider the case $\sigma(u_k; t) \neq 1$. In this case for $P_k(t)$, we obtain the expression:

$$d_t P_k(t) = \sigma(u_k; t) dw_k(t).$$

In this way we have that

$$d_t P_k^2(t) = \frac{\sigma^2(u_k; t)}{2} 2dt + 2P_k(t) \sigma(u_k; t) dw_k(t)$$

and

$$\begin{aligned} d_t \exp \left\{ -\frac{\alpha^2(b-a)}{2N} P_k^2(t) \right\} &= -\exp \left\{ -\frac{\alpha^2(b-a)}{2N} P_k^2(t) \right\} \times \\ &\times \frac{\alpha^2(b-a)}{2N} [\sigma^2(u_k; t) dt + 2P_k(t) \sigma(u_k; t) dw_k(t)] + \\ &+ \frac{\alpha^4(b-a)^2}{2N^2} P_k^2(t) \sigma^2(u_k; t) \cdot \exp \left\{ -\frac{\alpha^2(b-a)}{2N} P_k^2(t) \right\} dt. \end{aligned}$$

Therefore

$$\begin{aligned} d_t I_t(k; \alpha^2) &= \partial_t \mathbf{M} \left[\exp \left\{ -\frac{\alpha^2(b-a)}{2N} P_k^2(t) \right\} \right] = \\ &= -\mathbf{M} \left[\exp \left\{ -\frac{\alpha^2(b-a)}{2N} P_k^2(t) \right\} \frac{\alpha^2(b-a)}{2N} \sigma^2(u_k; t) \right] dt. \end{aligned}$$

By exploiting the possibility of differentiation with respect to the parameter α^2 , we arrive at the following equation:

$$\frac{\partial_t I_t(k; \alpha^2)}{\partial t} = -\frac{\alpha^2(b-a)}{2N} \sigma^2(u_k; t) I_t(k; \alpha^2) - \frac{\alpha^4(b-a)}{N} \sigma^2(u_k; t) \frac{\partial I_t(k; \alpha^2)}{\partial \alpha^2}. \quad (14)$$

We divide both members by $\sigma^2(u_k; t)$, and denote

$$\theta(t) = \int_0^t \sigma^2(u_k; \tau) d\tau$$

and we pass to the auxiliary equation

$$\frac{\partial I_\theta(k; \alpha^2)}{\partial \theta} = -\frac{\alpha^2(b-a)}{2N} I_\theta(k; \alpha^2) - \frac{\alpha^4(b-a)}{N} \frac{\partial I_\theta(k; \alpha^2)}{\partial \alpha^2}. \quad (15)$$

Equation (15) is a differential equation with constant coefficients stochastically equivalent to equation (13). Therefore its solution has the form:

$$I_\theta(k; \alpha^2) = \left(\frac{\alpha^2(b-a)}{2N} \theta + 1 \right)^{-1/2}.$$

and we obtain, the solution of the equation (14):

$$I_t(k; \alpha^2) = \left(\frac{\alpha^2(b-a)}{2N} \int_0^t \sigma^2(u_k; \tau) d\tau + 1 \right)^{-1/2}.$$

As a consequence we have that

$$\mathbf{M} [I_N(k; \alpha^2)] = \prod_{k=1}^N I_t(k; \alpha^2) = \prod_{k=1}^N \left(\frac{\alpha^2(b-a)}{2N} \int_0^t \sigma^2(u_k; \tau) d\tau + 1 \right)^{-1/2}. \quad (16)$$

In order to evaluate $I_1(a, b; t)$, we take the logarithm of (16):

$$\begin{aligned} \ln \mathbf{M} [I_N(k; \alpha^2)] &= \ln \prod_{k=1}^N \left(\frac{\alpha^2(b-a)}{2N} \int_0^t \sigma^2(u_k; \tau) d\tau + 1 \right)^{-1/2} = \\ &= -\frac{1}{2} \sum_{k=1}^N \ln \left(\frac{\alpha^2(b-a)}{2N} \int_0^t \sigma^2(u_k; \tau) d\tau + 1 \right). \end{aligned}$$

By using the series expansion of $\ln(x+1)$ we obtain that:

$$\begin{aligned} \ln \mathbf{M} [\tilde{I}_1(b; t)] &= -\frac{1}{2} \sum_{k=1}^N \left[\frac{\alpha^2(b-a)}{2N} \int_0^t \sigma^2(u_k; \tau) d\tau - \right. \\ &\quad \left. - \frac{1}{2} \frac{\alpha^4(b-a)^2}{4N^2} \left(\int_0^t \sigma^2(u_k; \tau) d\tau \right)^2 + O(N^{-3}) \right]. \end{aligned}$$

We calculate the limit for $N \rightarrow \infty$:

$$\begin{aligned} \lim_{N \rightarrow \infty} \ln \mathbf{M} [I_N(k, \alpha^2)] &= -\frac{1}{2} \lim_{N \rightarrow \infty} \sum_{k=1}^N \left[\frac{\alpha^2(b-a)}{2N} \int_0^t \sigma^2(u_k; \tau) d\tau - \right. \\ &\quad \left. -\frac{1}{2} \cdot \frac{\alpha^4(b-a)^2}{4N^2} \left(\int_0^t \sigma^2(u_k; \tau) d\tau \right)^2 + O(N^{-3}) \right] = \\ &= -\frac{1}{2} \lim_{N \rightarrow \infty} \sum_{k=1}^N \frac{\alpha^2(b-a)}{2N} \int_0^t \sigma^2(u_k; \tau) d\tau = -\frac{\alpha^2}{4} \int_a^b \left(\int_0^t \sigma^2(u; \tau) d\tau \right) du. \end{aligned}$$

By using the limit and passing to the anti-logarithm we prove that Lemma 5.2 holds

$$\mathbf{M} \left[\exp \left\{ -\frac{\alpha^2}{2} \int_a^b \eta^2(u; t) du \right\} \right] = \exp \left\{ -\frac{\alpha^2}{4} \int_a^b \left(\int_0^t \sigma^2(u; \tau) d\tau \right) du \right\}. \quad \square$$

5.3 Passing to the limiting process

The field $\{x_n(l; t); y_n(l; t)\}$ is defined by the model assumption (3) :

$$\eta(l_s; t) = \int_0^t \sigma(l_s; \tau) dw(\tau).$$

We change the model assumption

$$\tilde{\eta}(l_s; t) = \left(\frac{1}{2} \int_0^t \sigma^2(l_s; \tau) d\tau \right)^{1/2}$$

and consider the field $\{\hat{x}_n(l; t); \hat{y}_n(l; t)\}$ of the following form:

$$\begin{aligned} \hat{x}_n(l; t) &= \sum_{s=1}^n a(l_s) \cos \left[\sum_{j=1}^s \Delta(w(l_j)) \left(\frac{1}{2} \int_0^t \sigma^2(l_j; \tau) d\tau \right)^{1/2} \right] \Delta, \\ \hat{y}_n(l; t) &= \sum_{s=1}^n a(l_s) \sin \left[\sum_{j=1}^s \Delta(w(l_j)) \left(\frac{1}{2} \int_0^t \sigma^2(l_j; \tau) d\tau \right)^{1/2} \right] \Delta, \end{aligned} \quad (17)$$

where $\Delta(w(l_j))$ is the increment of the Wiener process on the interval $[l_j; l_{j+1}]$. This means that the variable t is not a random variable and from the analysis of the process on the flow of the σ -algebras $\mathfrak{F}_t(n) \oplus \mathfrak{F}_l$ it is possible to pass to the process defined on the flow of the σ -algebras $\mathfrak{F}(l)$, for all $t = \text{const}$.

Averaging with respect to t has already been carried out. We observe that the fields $\{x_n(l; t); y_n(l; t)\}$ and $\{\hat{x}_n(l; t); \hat{y}_n(l; t)\}$ are defined on different spaces. We consider the processes

$$z_2(k; t) = \exp \left\{ -i \sum_{j=1}^k \Delta w(l_j) \left(\frac{1}{2} \int_0^t \sigma^2(l_j; \tau) d\tau \right)^{1/2} \right\},$$

$$z_{n,2}(l; t) = \sum_{k=1}^n z_2(k; t) a(l_k) \cdot \Delta, \quad \Delta = O(n^{-1}).$$

By considering the Euler representation, the components of the fields (17) will take the form:

$$\begin{aligned} \hat{x}_n(l; t) &= \frac{1}{2} (z_{n,2}(l; t) + z_{n,2}^*(l; t)), \\ \hat{y}_n(l; t) &= \frac{i}{2} (z_{n,2}(l; t) - z_{n,2}^*(l; t)). \end{aligned}$$

We construct now the characteristic functions: $g_n(\alpha; \beta; t)$ for the field $\{x_n(l; t); y_n(l; t)\}$ and $\hat{g}_n(\alpha; \beta; t)$ for the field $\{\hat{x}_n(l; t); \hat{y}_n(l; t)\}$:

$$\begin{aligned} g_n(\alpha; \beta; t) &= \mathbf{M} \left[\exp \left\{ \frac{i}{2} (\alpha + i\beta) z_{n,1}(l; t) + \frac{i}{2} (\alpha - i\beta) z_{n,1}^*(l; t) \right\} \right], \\ \hat{g}_n(\alpha; \beta; t) &= \mathbf{M} \left[\exp \left\{ \frac{i}{2} (\alpha + i\beta) z_{n,2}(l; t) + \frac{i}{2} (\alpha - i\beta) z_{n,2}^*(l; t) \right\} \right]. \end{aligned}$$

For the continuation of the research the next lemma is necessary.

Lemma 5.3 *Under the model assumptions for the random fields $\{x_n(l; t); y_n(l; t)\}$ and $\{\hat{x}_n(l; t); \hat{y}_n(l; t)\}$ and for fixed integer m there exists a number n' such that for all $n > n'$ the following relationships hold:*

$$\mathbf{M}[z_{,2}^m(l; t)] = \mathbf{M}[z_{,2}^{*m}(l; t)] = \mathbf{M}[z_{,1}^m(l; t)] = \mathbf{M}[z_{,1}^{*m}(l; t)]. \quad (18)$$

Proof. We calculate all increments in the series of the equalities (18) by having in mind the lemmas. We introduce the following notation $\tilde{\eta}(\theta, t) =$

$\left(\frac{1}{2} \int_0^t \sigma^2(\theta, \tau) d\tau\right)^{1/2}$. We have that

$$\begin{aligned}
\mathbf{M}[z_{,2}^m(l; t)] &= m! \int_0^l a(u_1) du_1 \mathbf{M} \left[\exp \left\{ -i m \int_0^{u_1} \tilde{\eta}(\theta_1, t) dw(\theta_1) \right\} \right] \times \\
&\times \dots \times \int_{u_m}^l a(u_m) du_m \mathbf{M} \left[\exp \left\{ -i \int_{u_{m-1}}^{u_m} \tilde{\eta}(\theta_m, t) dw(\theta_m) \right\} \right] = \\
&= m! \left(\int_0^l a(u_1) du_1 \exp \left\{ -\frac{m^2}{2} \int_0^{u_1} \tilde{\eta}^2(\theta_1, t) d\theta_1 \right\} \right) \times \\
&\times \dots \times \left(\int_{u_m}^l a(u_m) du_m \exp \left\{ -\frac{1}{2} \int_{u_{m-1}}^{u_m} \tilde{\eta}^2(\theta_m, t) d\theta_m \right\} \right) = \\
&= m! \left(\int_0^l a(u_1) du_1 \exp \left\{ -\frac{m^2}{2} \int_0^{u_1} \left(\frac{1}{2} \int_0^t \sigma^2(\theta_1, \tau) d\tau \right) d\theta_1 \right\} \right) \times \dots \times \\
&\times \left(\int_{u_m}^l a(u_m) du_m \exp \left\{ -\frac{1}{2} \int_{u_{m-1}}^{u_m} \left(\frac{1}{2} \int_0^t \sigma^2(\theta_m, \tau) d\tau \right) d\theta_m \right\} \right) = \mathbf{M}[z_{,1}^m(l; t)]
\end{aligned}$$

Then, in force of (5) and (7) we have that:

$$\mathbf{M}[z_{,1}^m(l; t)] = \mathbf{M}[z_{,1}^{*m}(l; t)].$$

In this way we obtain the confirmation of Lemma. \square

Lemma 5.4 *The characteristic functions of fields $\{x_n(l; t); y_n(l; t)\}$ and $\{\hat{x}_n(l; t); \hat{y}_n(l; t)\}$ for $n \rightarrow \infty$ coincide for all l and t .*

Proof. The proof is based on the coincidence of the representations for the characteristic functions $g_n(\alpha, \beta, t)$ and $\hat{g}_n(\alpha, \beta, t)$ by means of the Maclaurin expansion (inside the mean) with respect to $z_{,1}(l; t)$ and $z_{,1}^*(l; t)$, and $z_{,2}(l; t)$, $z_{,2}^*(l; t)$ respectively and also on the conclusions of Lemma 5.3. \square

Lemma 5.4 permits us to pass to the study of the limit behavior of the field $\{\hat{x}_n(l; t); \hat{y}_n(l; t)\}$ for $n \rightarrow \infty$ exclusively.

Theorem 5.1 *Let us assume that for the field $\{x_n(l; t); y_n(l; t)\}$ the model assumptions (17) are satisfied:*

$$\begin{aligned}
\hat{x}_n(l; t) &= \sum_{s=1}^n a(l_s) \cos \left[\sum_{j=1}^s \Delta(w(l_j)) \left(\frac{1}{2} \int_0^t \sigma^2(l_j; \tau) d\tau \right)^{1/2} \right] \Delta, \\
\hat{y}_n(l; t) &= \sum_{s=1}^n a(l_s) \sin \left[\sum_{j=1}^s \Delta(w(l_j)) \left(\frac{1}{2} \int_0^t \sigma^2(l_j; \tau) d\tau \right)^{1/2} \right] \Delta,
\end{aligned}$$

and assume that the field $\{x(l; t); y(l; t)\}$ is defined in the following way

$$\begin{aligned} x(l; t) &= \int_0^l a(u) \cos \left[\int_0^u \frac{1}{2} \left(\int_0^t \sigma^2(\theta, \tau) d\tau \right) dw(\theta) \right] du, \\ y(l; t) &= \int_0^l a(u) \sin \left[\int_0^u \frac{1}{2} \left(\int_0^t \sigma^2(\theta, \tau) d\tau \right) dw(\theta) \right] du. \end{aligned} \quad (19)$$

Under these conditions the characteristic functions of the processes $\{x(l; t); y(l; t)\}$ and $\{x_n(l; t); y_n(l; t)\}$ coincide.

Proof. The comparison of the characteristic functions for $\{\hat{x}_n(l; t), \hat{y}_n(l; t)\}$ and $\{x(l; t), y(l; t)\}$ for all values of $t \in [0, T]$, for $n \rightarrow \infty$ leads to the proof of the theorem. \square

Theorem 5.2 *The stochastic process $\{x(l; t); y(l; t)\}$ is the solution to the Cauchy problem for the Ito stochastic differential equations:*

$$\begin{aligned} d_l p(l; t) &= \left[p(l; t) \frac{\partial}{\partial l} \ln a(l) - \frac{p(l; t)}{4} \int_0^t \sigma^2(l; \tau) d\tau \right] dl - \left(\frac{1}{2} \int_0^t \sigma^2(l; \tau) d\tau \right)^{0,5} q(l; t) dw(l), \\ d_l q(l; t) &= \left[q(l; t) \frac{\partial}{\partial l} \ln a(l) - \frac{q(l; t)}{4} \int_0^t \sigma^2(l; \tau) d\tau \right] dl + \left(\frac{1}{2} \int_0^t \sigma^2(l; \tau) d\tau \right)^{0,5} p(l; t) dw(l), \\ d_l x(l; t) &= q(l; t) dl, \quad d_l y(l; t) = p(l; t) dl, \end{aligned} \quad (20)$$

satisfying the boundary conditions

$$x(0; t) = 0, \quad y(0; t) = 0, \quad p(0; t) = a(0), \quad q(0; t) = 0.$$

Proof. We differentiate $x(l; t)$ and $y(l; t)$ in (19) with respect to l :

$$\frac{\partial x(l; t)}{\partial l} = a(l) \sin \left[\int_0^l \frac{1}{2} \left(\int_0^t \sigma^2(\theta, \tau) d\tau \right) dw(\theta) \right] = q(l; t), \quad (21)$$

$$\frac{\partial y(l; t)}{\partial l} = -a(l) \cos \left[\int_0^l \frac{1}{2} \left(\int_0^t \sigma^2(\theta, \tau) d\tau \right) dw(\theta) \right] = p(l; t). \quad (22)$$

The obtained expressions are now differentiated by Ito formula with respect to the variable l :

$$\begin{aligned} d_l \left(\frac{\partial x(l; t)}{\partial l} \right) &= \frac{1}{a(l)} \frac{\partial a(l)}{\partial l} a(l) \cos \left[\int_0^l \frac{1}{2} \left(\int_0^t \sigma^2(\theta, \tau) d\tau \right) dw(\theta) \right] dl - \\ &- a(l) \sin \left[\int_0^l \frac{1}{2} \left(\int_0^t \sigma^2(\theta, \tau) d\tau \right) dw(\theta) \right] \cdot \frac{1}{2} \left(\int_0^t \sigma^2(\theta, \tau) d\tau \right) dw(l) - \\ &- \frac{1}{2} \cos \left[\int_0^l \frac{1}{2} \left(\int_0^t \sigma^2(\theta, \tau) d\tau \right) dw(\theta) \right] \cdot \left(\frac{1}{2} \int_0^t \sigma^2(\theta, \tau) d\tau \right)^2 dl. \end{aligned}$$

Taking into account (21) and (22) we obtain the last equation of system (20). In a similar way we get the second expression of the system. The functions $x(l; t)$, $y(l; t)$, $p(l; t)$, $q(l; t)$ defined by (19), (21) and (22) satisfy the given initial conditions. \square

Within the framework of the given formulation (L constant) we have found $F_t(x; y; L)$ for different values of t .

Theorem 5.3 *The distribution function of the process $\{x(l; t); y(l; t)\}$ can be obtained by integrating with respect to the variables p and q the Kolmogorov equation of the system (20).*

Proof. After the enlargement of the space obtained by introducing the new variables p and q , the compound process $\{x(l; t); y(l; t); p(l; t); q(l; t)\}$ becomes a Markov process. This means that it is possible to obtain a Kolmogorov equation for the density function $\rho(x, y, p, q, l, t)$ and then by integrating with respect to p and q infer the density function of the distribution $\rho(x, y, l, t)$ for all l and t . \square

Theorem 5.4 *The distribution function of the original process $\{x_n(l; t); y_n(l; t)\}$ under the model conditions (4) coincide with the distribution function of the Markov process $\{x(l; t); y(l; t)\}$ (19).*

Proof. The proof is based on the conclusions of Theorem 5.1 and Theorem 5.2. \square

Remark 5.1 *The character of the analysis doesn't substantially changes when, for example $a = a(l, t)$ (vibrating chain), $\sigma(l; t)$ is a non anticipating measurable random function with respect to independent flows of σ -algebras governed by independent Wiener processes $w(l)$ and $w(t)$.*

In this way we arrive at a coherent representation of distribution: the parameter t defines also the structure of the chain.

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